Steady-state power-law creep in inclusion matrix composite-materials

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STEADY-STATE POWER-LAW CREEP IN "INCLUSION MATRIX" COMPOSITE MATERIALS

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Abstract—This work is devoted to the prediction of the constitutive steady-state creep behavior of matrix inclusion composites. Both phases are characterized by power-law constitutive equations. The three phase model is extended to viscoplastic equations. If both phases have the same strain rate sensitivity, the effective behavior of the composite is characterized by an effective prefactor. If not, an effective strain rate sensitivity is defined, which is a function of the applied strain rate and of the volume fraction of the phases. All the results are compared with the classical self-consistent ones. A limit case which may be related to the grain boundary sliding accommodated by intragranular power-law creep is also studied.

Résumé—Ce travail est consacré à la prédiction du comportement en fluage stationnaire de matériaux composites de type inclusion/matrice pour lesquels chacune des phases est caractérisée par une loi puissance. A cet effet, une extension du modèle "trois phases" à des comportements viscoplastiques est présentée. Si les phases ont le même exposant (sensibilité à la vitesse de déformation), on définit un préfacteur effectif. Sinon, on définit un exposant effectif qui dépend de la vitesse de déformation appliquée et des concentrations des phases. Un cas limite est également abordé, qui peut être comparé à de précédents modèles de glissement aux joints accommodé par du fluage intragranulaire.

1. INTRODUCTION

High temperature deformation of engineering materials is often described by a Norton–Hoff law relating stress state to strain rate via a power-law expression. Most of these materials are multiphase materials, where individual phases may have different behavior. A question that often arises in such materials is: if individual phases have different constitutive relations, what is the constitutive relation of the composite material?

This study is interested in predicting such effective behaviors. The focus is mainly on composite materials for which it is possible to distinguish a phase (which will be called "inclusion") embedded in a continuous phase (which will be called "matrix"). It is particularly interesting to explore the conditions (in terms of phase volume fraction or applied strain rate) for which the strain rates are more pronounced in one or the other phase and consequently, to determine the respective influences of each phase on the effective behavior.

Theories for determining the overall linear-elastic properties of two-phase composites by means of rigorous bounds or estimates are well-developed. Many investigations have also been performed for materials with a plastically deforming matrix containing elastic or rigid reinforcements. Fewer studies have focused on composites in which both phases deform inelastically. In this field, steady-state creep is often described by a Norton–Hoff law relating stress state to strain rate via a power-law expression. When a composite includes two phases, both characterized by power-law constitutive equations, different types of approach may be used to describe the effective behavior. One can first distinguish numerical predictions based on finite element computations [1] and analytical ones. Concerning the latter, if the field is restricted to random materials, bounds [2, 3] or self-consistent estimates may be obtained [2, 4]. On the one hand, first order bounds (analogous to "Reuss" and "Voigt" bounds in elasticity) are far apart whenever the moduli of the constituent phases are quite different. The use of second order bounds (analogous to "Hashin & Shtrikman" bounds in elasticity) leads to a limited improvement [5]. On the other hand, self-consistent estimates are implicitly related to perfectly disordered materials. This excludes a realistic description of "inclusion-matrix" type composites where the specific morphology (i.e. separate zones embedded in a continuous phase) must be taken into account. The

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use of the three-phase model [6] allowed the study of such materials for elastic behaviors. This model was extended to elastoplastic behaviors [7], opening the way to the study of non-linear materials.

The aim of this paper is to derive an extension of the "three-phase" model for power-law constitutive equations. The new formulation is presented in Section 2. The different levels of approximation are described. A general tensorial description is first presented. A simplified scalar notation is then adopted, which does not restrict the validity of the study. Section 3 is devoted to the results of the simulations. Different cases are successively studied. Section 3.1 (respectively Section 3.2) corresponds to the case of identical (respectively different) strain rate sensitivity for both phases. Analytical results for the case of very small and very high strain rates are presented. An analogy is made with a ductile matrix containing respectively rigid inclusions or very soft ones. In the general case, numerical results are given for the effective strain rate sensitivity and the effective prefactor of the overall power law expression. All these results are discussed by comparison with the classical self-consistent results. Finally, Section 3.3 is devoted to the study of the special configuration of very low concentration of matrix, which may be compared to previous results concerning the deformation occurring by grain-boundary sliding accommodated by intragranular power-law creep.

2. DESCRIPTION AND FORMULATION OF THE MODEL

2.1. Constitutive relations

Let us consider a two-phase composite constituted of two isotropic incompressible phases creeping according to a power law. Constitutive equations for phase α ($\alpha = 1, 2$) have the form

$$\underset{\sim}{s}(x) = 2A_\alpha[\dot{\epsilon}_{eq}(x)]^{m_\alpha - 1}\underset{\sim}{\dot{e}}(x) \tag{1}$$

$\underset{\sim}{s}(x)$ and $\underset{\sim}{\dot{e}}(x)$ are respectively the stress and strain rate deviators at current position x within phase α. A_α and m_α are measured constant. $\dot{\epsilon}_{eq}(x)$ is the equivalent strain rate at x, defined by

$$\dot{\epsilon}_{eq}(x) = (\tfrac{2}{3}\underset{\sim}{\dot{e}}(x):\underset{\sim}{\dot{e}}(x))^{1/2}. \tag{2}$$

The constitutive relation (1) may be written

$$\underset{\sim}{s}(x) = 2\mu(x)\underset{\sim}{\dot{e}}(x) \tag{3}$$

where the secant modulus $\mu(x)$ is equal to

$$\mu(x) = A_\alpha[\dot{\epsilon}_{eq}(x)]^{m_\alpha - 1}, \quad x \in \alpha. \tag{4}$$

Denoting $\langle f \rangle_\alpha$ the spatial average over phase α of any function f, the average value of f over a representative elementary volume of the composite is

$$\langle f \rangle = c_1\langle f \rangle_1 + c_2\langle f \rangle_2. \tag{5}$$

c_1 and c_2 are the phase concentrations with $c_1 + c_2 = 1$. Spatial averaging of (1) over phase α gives

$$\langle \underset{\sim}{s} \rangle_\alpha = 2A_\alpha\langle \dot{\epsilon}_{eq}^{m_\alpha - 1}\underset{\sim}{\dot{e}} \rangle_\alpha. \tag{6}$$

Let us define the secant moduli $\bar{\mu}_\alpha$ and $\bar{\mu}$ so that

$$\langle \underset{\sim}{s} \rangle_\alpha = 2\bar{\mu}_\alpha\langle \underset{\sim}{\dot{e}} \rangle_\alpha \quad \text{and} \quad \langle \underset{\sim}{s} \rangle = 2\bar{\mu}\langle \underset{\sim}{\dot{e}} \rangle. \tag{7}$$

If the phases are arranged in a manner that is statistically uniform and isotropic, the so-defined secant moduli $\bar{\mu}_\alpha$ and $\bar{\mu}$ can be assumed to be respectively scalar functions of the equivalent strain rates $(\langle \underset{\sim}{\dot{e}} \rangle_\alpha)_{eq}$ and $(\langle \underset{\sim}{\dot{e}} \rangle)_{eq}$ (though this way, the possible role of the third invariant in the overall behavior is then neglected). The homogenization problem consists then in finding the function $\bar{\mu}$ or alternatively in determining the variation of $\langle \underset{\sim}{s} \rangle$ as a function of $\langle \underset{\sim}{\dot{e}} \rangle$.

2.2. The three phase model

Consider now a two-phase composite of "matrix inclusion" type: one phase is continuous throughout the composite whereas the other is constituted of discrete inclusion. In the following, phase 1 denotes the matrix and phase 2 denotes the inclusions. The inclusions are assumed randomly distributed within the matrix. Perfect bonding is assumed at the interfaces. In order to derive the effective characteristics of such a medium, let us refer to the scheme of the three phase model [7]. We consider a composite sphere constituted of a spherical core with the characteristics of the phase 2 and of a concentric shell with the characteristics of the phase 1 (Fig. 1). This composite sphere is embedded in a continuum subjected to uniform stress $\underset{\sim}{s}_0$ or strain rate $\underset{\sim}{\dot{e}}_0$ at infinity. This continuum represents the homogeneous equivalent medium, whose unknown characteristics define the effective behavior of the composite.

Strictly speaking, the three phase model cannot be applied at this stage, because the secant moduli defined in equation (4) are varying even when the prefactors A_α are constant in the corresponding part of the composite inclusion. A further approximation will be therefore to assume that the secant moduli are approximately constant within each homogeneous part of the composite sphere and equal to the moduli $\bar{\mu}_\alpha$ defined in equation (7).

Moreover, the three phase formulation requires an explicit expression of these moduli. With $\dot{\epsilon}_\alpha$ denoting the equivalent average strain rate $(\langle \underset{\sim}{\dot{e}} \rangle_\alpha)_{eq}$, the following expression is chosen

$$\mu(x) = \bar{\mu}_\alpha \approx A_\alpha(\dot{\epsilon}_\alpha)^{m_\alpha - 1}, \quad x \in \alpha. \tag{8}$$

It must be stressed at this point that the approximation $\langle \dot{\epsilon}_{eq}^{m_\alpha - 1}(x) \rangle \approx \langle \dot{\epsilon}_{eq}(x) \rangle^{m_\alpha - 1}$ is implicitly made. This approximation leads to an overestimation of $\bar{\mu}_\alpha$ because m_α is lower than 1. As a consequence, the effective modulus $\bar{\mu}$ will be overestimated by the procedure and the overestimation will be all the more crude since the non-linearities will be important.

Following the extension of Hervé and Zaoui [7] to non-linear materials, it is now assumed that the average stress tensor and the average strain rate tensor within phase α are obtained by the classical three phase equations applied to an elastic composite sphere [see equations (A1)–(A4) of the Appendix]

$$\langle \dot{\underline{e}} \rangle_\alpha = a_\alpha \langle \dot{\underline{e}} \rangle = a_\alpha \dot{\underline{e}}_0 \quad \text{and} \quad \langle \underline{s} \rangle_\alpha = b_\alpha \langle \underline{s} \rangle = b_\alpha \underline{s}_0. \tag{9}$$

a_α and b_α are functions of $\bar{\mu}$, $\bar{\mu}_1$, $\bar{\mu}_2$, c_1, c_2. Equations (9) combined with average conditions and approximate expressions of $\bar{\mu}_\alpha$ lead to a second order equation for the ratio $\bar{\mu}/\mu_1$ and hence to the solution of the homogenization problem. a_α, b_α and the coefficients of this second order equation are given in the Appendix [equations (A1), (A2) and (A6–8)].

2.3. Simplified formulation

The deviatoric stress corresponding to prescribed uniform strain rate conditions $\dot{\underline{e}}_0$ may be easily deduced from equation (7)

$$\underline{s}_0 = 2\bar{\mu}\dot{\underline{e}}_0. \tag{10}$$

Without restricting the generality of the study, it is worth using the following equivalent stress and strain rate

$$\sigma_0 = (\tfrac{3}{2}\underline{s}_0 : \underline{s}_0)^{1/2},$$

$$\dot{\epsilon}_0 = (\tfrac{2}{3}\dot{\underline{e}}_0 : \dot{\underline{e}}_0)^{1/2} = \frac{\sigma_0}{3\bar{\mu}}. \tag{11}$$

Using the linearity of equations (9), the equivalent stress and strain rate in phase α may be expressed as

$$[\langle \dot{\underline{e}} \rangle]_{\text{eq}} = \dot{\epsilon}_\alpha = a_\alpha \frac{\sigma_0}{3\bar{\mu}}, \quad [\langle \bar{s} \rangle_\alpha]_{\text{eq}} = \sigma_\alpha = b_\alpha \sigma_0. \tag{12}$$

Moreover, combining relations (7), (9) and (10) leads to the following relation

$$b_\alpha \bar{\mu} = a_\alpha \bar{\mu}_\alpha. \tag{13}$$

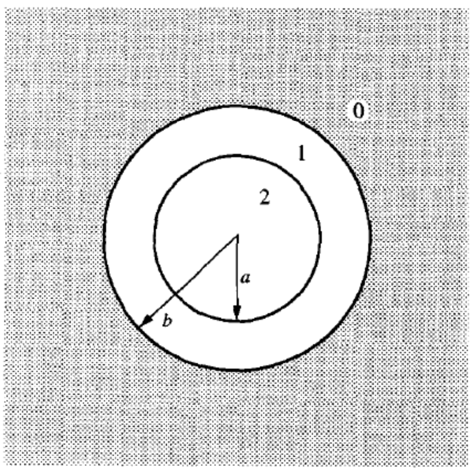


Fig. 1. The three phase model. 0: Homogeneous equivalent medium; 1: matrix and 2: inclusion.

It is then possible to write from equation (12) the constitutive scalar equation for each phase

$$\sigma_\alpha = 3\bar{\mu}_\alpha \dot{\epsilon}_\alpha. \tag{14}$$

It follows easily from the following equations (cf. Appendix)

$$c_1 a_1 + c_2 a_2 = 1 \quad \text{and} \quad c_1 b_1 + c_2 b_2 = 1 \tag{15}$$

that

$$\begin{cases} c_1 \bar{\mu}_1 \dot{\epsilon}_1 + c_2 \bar{\mu}_2 \dot{\epsilon}_2 = \bar{\mu}_1 h\left(c_1, c_2, \dfrac{\bar{\mu}_2}{\bar{\mu}_1}\right)\dot{\epsilon}_0 \\ c_1 \dot{\epsilon}_1 + c_2 \dot{\epsilon}_2 = \dot{\epsilon}_0 \end{cases} \tag{16}$$

where $h(c_1, c_2, \bar{\mu}_1/\bar{\mu}_2) = \bar{\mu}/\bar{\mu}_1$ denotes the positive root of the second order equation (A5) given in the Appendix.

$\dot{\epsilon}_0$ being given, the resolution of equation (16) leads to the equivalent strain rate in each phase $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ and then leads to the homogenization problem solution $\bar{\mu}(\dot{\epsilon}_0)$.

The solution is characteristic of the three phase configuration through the solution $h(c_1, c_2, \bar{\mu}_2/\bar{\mu}_1)$. Replacing h by the solution of the following second order equation

$$3\left(\frac{\bar{\mu}}{\bar{\mu}_1}\right)^2 + \left[\frac{\bar{\mu}_2}{\bar{\mu}_1}(2c_1 - 3c_2) + (2c_2 - 3c_1)\right]\left(\frac{\bar{\mu}}{\bar{\mu}_1}\right) - 2\frac{\bar{\mu}_2}{\bar{\mu}_1} = 0 \tag{17}$$

gives the viscoplastic extension of the classical self-consistent scheme [4].

3. EFFECTIVE BEHAVIOR

The two phases of the composite material are characterized by

$$\sigma_\alpha = 3A_\alpha \dot{\epsilon}_\alpha^{m_\alpha}, \quad \alpha = 1, 2. \tag{18}$$

Various cases are now considered depending on the values of the prefactors A_α, of the strain rate sensitivities m_α and of the phase volume fractions.

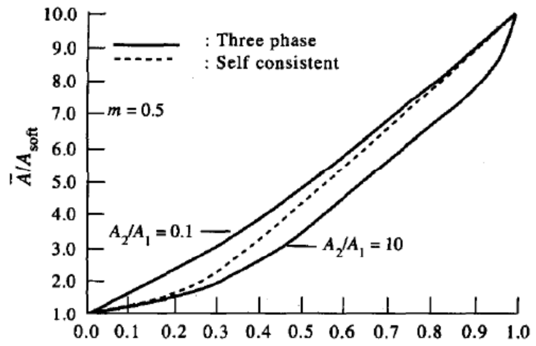
3.1. Equal strain rate sensitivities

The constitutive laws of the two phases differ only in the prefactor. The hard phase refers to the highest value of A_α and the soft phase refers to the lowest value of A_α. The constitutive law of the composite is characterized by the exponent $m = m_1 = m_2$ and by an effective prefactor $\bar{A}$

$$\sigma_0 = 3\bar{A}\dot{\epsilon}_0^m. \tag{19}$$

With the notations $c = c_2$, $x = \dot{\epsilon}_2/\dot{\epsilon}_0$ and $y = \dot{\epsilon}_1/\dot{\epsilon}_0$, the system in equation (16) may be written

$$\begin{cases} cA_2 x^m + (1-c)A_1 y^m = A_1 y^{m-1} h\left(c, \dfrac{A_2}{A_1}\left(\dfrac{x}{y}\right)^{m-1}\right) \\ cx + (1-c)y = 1 \end{cases} \tag{20}$$



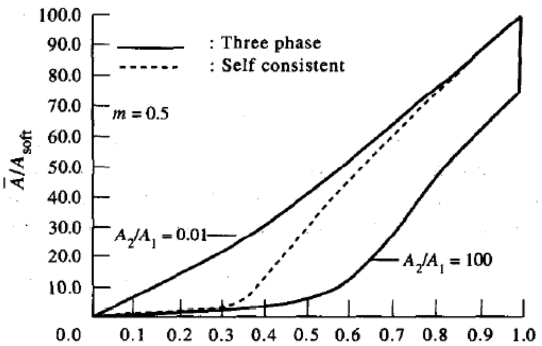


Fig. 2. The normalized effective prefactor $\bar{A}/A_1$ vs the hard phase concentration ($m = m_1 = m_2 = 0.5$). Self-consistent and three phase model.

x and y are derived from the resolution of equation (20). $\bar{A}$ is then given by

$$\bar{A} = A_1 y^{m-1} h\left(c, \frac{A_2}{A_1}\left(\frac{x}{y}\right)^{m-1}\right). \tag{21}$$

The ratio $\bar{A}/A_{soft}$ is plotted on Fig. 2 as a function of the hard phase concentration for $m = 0.5$ and for different values of the ratio A_2/A_1. Both situations with a matrix stiffer or softer than the inclusions are considered. Classical self-consistent values obtained from equation (17) are also reported. Self-consistent and three phase models give similar results for a relatively high concentration of matrix (depending on the ratio A_2/A_1). However, it may be seen that taking into account the connectedness of phase 1 gives very strong differences out of this range. These differences are all the more important since the ratio A_2/A_1 is large. Recent works [5, 8, 9] have shown that the use of adequate variational principles lead to the determination of a second order upper bound (Hashin and Shtrikman type) for $\bar{A}/A_{soft}$. Results are not reported here but they indicate clearly that this bound is violated by our present approach for concentrations exceeding a certain value c (depending on m and on the ratio A_2/A_1). Further investigations have to be carried out in our next work, particularly in order to verify if our present overestimations lead to acceptable (or not) differences with experimental data.

3.2. Different strain rate sensitivities

• *Normalization*—m_1 and m_2 are now different with, for example, $m_1 > m_2$. $m_1 \neq m_2$ ensures that the curves representing $\sigma_\alpha(\dot{\epsilon}_\alpha)$ display a common point $(\dot{\epsilon}_c, \sigma_c)$ such that $\dot{\epsilon}_1 = \dot{\epsilon}_2 = \dot{\epsilon}_c$ and $\sigma_1 = \sigma_2 = \sigma_c$ (Fig. 3), where

$$\dot{\epsilon}_c = \left(\frac{A_2}{A_1}\right)^{1/(m_1 - m_2)}$$

and

$$\sigma_c = \left[\frac{(3A_2)\dfrac{1}{m_2}}{(3A_1)\dfrac{1}{m_1}}\right]^{m_1 m_2/(m_1 - m_2)} . \tag{22}$$

It is therefore possible to write the constitutive relations of equation (18) using the normalized variables σ_α/σ_c and $\dot{\epsilon}_\alpha/\dot{\epsilon}_c$ so that

$$\left(\frac{\sigma_1}{\sigma_c}\right) = \left(\frac{\dot{\epsilon}_1}{\dot{\epsilon}_c}\right)^{m_1} \quad \text{and} \quad \left(\frac{\sigma_2}{\sigma_c}\right) = \left(\frac{\dot{\epsilon}_2}{\dot{\epsilon}_c}\right)^{m_2}. \tag{23}$$

• *Asymptotic solutions*—numerical calculation is generally needed to derive solutions. However, solutions at $\dot{\epsilon}_0 \approx \infty$ and at $\dot{\epsilon}_0 \approx 0$ are readily obtainable

*$\dot{\epsilon}_0 \approx \infty$: if the strain rates in each phase are of the same order of magnitude

$$\dot{\epsilon}_1^{m_1 - 1} \gg \dot{\epsilon}_2^{m_2 - 1} \quad \text{so that} \quad \bar{\mu}_1 \gg \bar{\mu}_2 \text{ and } \sigma_1 \gg \sigma_2.$$

Under these conditions, the asymptotic solution for $\bar{\mu}$ corresponds to the solution for the effective modulus of a viscoplastic matrix containing very soft inclusions with a concentration c. Elimination of $\bar{\mu}/\bar{\mu}_1$ in equations (16) and (A1) leads to the effective constitutive law

$$\left(\frac{\sigma_0}{\sigma_c}\right) = (1-c)^{1-m_1}[h_\infty(c)]^{m_1}\left(\frac{\dot{\epsilon}_0}{\dot{\epsilon}_c}\right)^{m_1}. \tag{24}$$

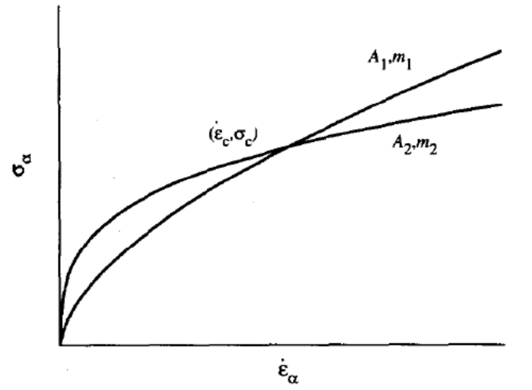


Fig. 3. Evidence of a common point for strain rate/stress constitutive equations if $m_1 \neq m_2$.

$h_\infty(c)$ is the solution of the second order equation (A5) when $\bar{\mu}_1 \gg \bar{\mu}_2$. It is worth noticing that the assumption $\dot{\epsilon}_1 \gg \dot{\epsilon}_2$ leads to the same result, whereas the assumption $\dot{\epsilon}_1 \ll \dot{\epsilon}_2$ would lead to a non-physical solution.

*$\dot{\epsilon}_0 \approx 0$: the stresses are assumed of the same order of magnitude in each phase so that

$$\dot{\epsilon}_1 \gg \dot{\epsilon}_1 \quad \text{and} \quad \bar{\mu}_1 \ll \bar{\mu}_2.$$

The asymptotic solution for $\bar{\mu}$ corresponds to the case of a viscoplastic matrix containing very hard inclusions with a concentration c. $\dot{\epsilon}_0$ is then given by $\dot{\epsilon}_0 \approx (1-c)\dot{\epsilon}_1$ and $\bar{\mu}$ by $\bar{\mu} = A_1 h_0(c)\dot{\epsilon}_1^{m_1-1}$ where $h_0(c)$ is the solution of the second order equation (A5) when $\bar{\mu}_1 \ll \bar{\mu}_2$. The effective constitutive law is then

$$\left(\frac{\sigma_0}{\sigma_c}\right) = (1-c)^{1-m_1} h_0(c)\left(\frac{\dot{\epsilon}_0}{\dot{\epsilon}_c}\right)^{m_1}. \tag{25}$$

The two macroscopic constitutive laws of equations (24) and (25) show respectively that when the macroscopic strain rate becomes very large (respectively very low), the effective creep behavior of the two-phase composite becomes analogous to the behavior of an incompressible viscoplastic matrix containing very soft (respectively rigid) inclusions. The range of validity of this analogy depends of course on the values of m_1, m_2, A_1 and A_2. As indicated by equations (24) and (25), this behavior is always described by a power law characterized by the strain rate sensitivity of the matrix phase and by a prefactor which is a function of the solution of the second order equation (A5) in the two respective limits. Moreover, it is possible to show that the opposite choice $m_2 > m_1$ does not affect this conclusion.

These results may be compared to those given by the classical self-consistent model, which leads to quite different behaviors [4]. In this latter case, the overall creep behavior is governed by one or the other phase, depending on the phase volume fraction. More precisely, the value of the exponent of the macroscopic power law is related to the two "quasi percolation thresholds" $c = 0.4$ and $c = 0.6$ which are inherent to the classical self-consistent scheme. In the case of rigid inclusions for example, the effective exponent is equal to m_1 if $c_2 < 0.4$ and is equal to m_2 if $c_2 > 0.4$.

The normalized reference stress appearing in equation (25) is plotted on Fig. 4 vs the inclusion volume fraction for different values of m_1. It corresponds to the case of rigid inclusions embedded in a viscoplastic matrix. A comparison with the classical self-consistent results [4] is possible in the range $0 < c < 0.4$. The deformation resistance derived from the three phase model is shown to be lower than the one derived from the self-consistent scheme. It confirms that the three phase model takes into account the connectedness of the viscoplastic matrix.

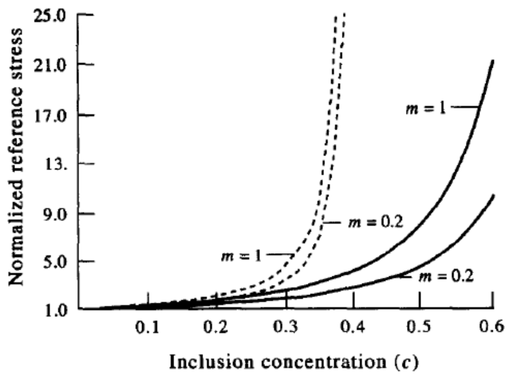


Fig. 4. Normalized reference stress vs inclusion concentration. Rigid inclusions embedded in a viscoplastic matrix. Self-consistent results (dashed rule) and three phase ones (solid rule).

Note that the curves of Fig. 4 are comparable (at least for values of c which are not too large) with previous results derived by a differential self-consistent scheme [10, 11]. However, it is more difficult to give a well defined topological signification to this differential scheme.

• *General solutions*—Fig. 5 shows a typical curve $(\sigma_0/\sigma_c, \dot{\epsilon}_0/\dot{\epsilon}_c)$ obtained by solving the system of equation (16) for different values of $(\dot{\epsilon}_0/\epsilon_c)$. When the level of the macroscopic strain rate varies between the two limits studied above, the behavior goes through a transition which can be better quantified by defining an *effective strain rate sensitivity* for the overall creep of the composite

$$m_{\text{eff}} = \frac{\mathrm{d}\log\sigma_0}{\mathrm{d}\log\dot{\epsilon}_0} = \frac{\mathrm{d}\sigma_0}{\mathrm{d}\dot{\epsilon}_0}\frac{\dot{\epsilon}_0}{\sigma_0}. \tag{26}$$

A specific example of the variation of m_{eff} with the normalized strain rate $(\dot{\epsilon}_0/\dot{\epsilon}_c)$ for $m_1 = 0.2$ and $m_2 = 0.1$ is shown in Fig. 6. Figure 6(a) corresponds to the classical self-consistent scheme. Figure 6(b) corresponds to the three phase model where the matrix phase is characterized by $m_1 = 0.2$ and where the inclusion phase is characterized by $m_2 = 0.1$. It may be seen that except for an amount of inclusions lower than 20%, the two schemes lead to quite different behaviors. In particular, as already noticed, the asymptotic behaviors are totally different for the two models. For the self-consistent model, the transition region governed by the percolation thresholds $c = 40$ and 60% appears clearly. For the three phase model the effective strain rate sensitivity tends to m_2 in a certain range but it always returns to m_1 at very high and very low macroscopic strain rates even for high concentrations of inclusions (although it is not clear in Fig. 6 for $c = 95\%$).

"Self-consistent" curves and "three phase" ones have a common point at a same concentration for $(\dot{\epsilon}_0/\dot{\epsilon}_c) = 1$: the two secant moduli are the same for both phases. m_{eff} is always higher for the three phase model than for the self-consistent one for any

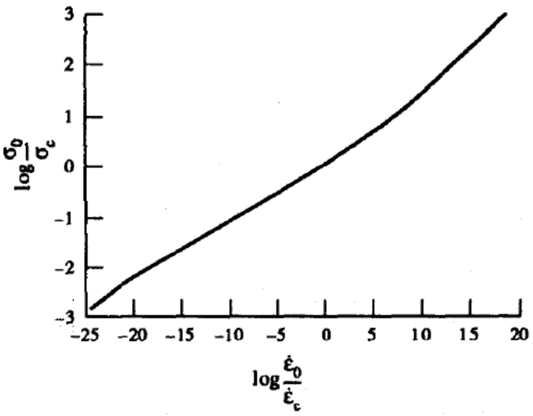


Fig. 5. Three phase model: $\sigma_0/\sigma_c = f(\dot{\epsilon}_0/\dot{\epsilon}_c)$ in log/log scale. $m_1 = 0.5$, $m_2 = 0.2$ and $c = 80\%$.

other value of the normalized macroscopic strain rate.

The evolution of m_{eff} displays a local minimum for very high concentrations of inclusions ($c > 90\%$). This feature is studied in the next section.

3.3. *"Intergranular" localization of the deformation*

This section deals with an interesting configuration of the three phase model for which the volume fraction of the matrix phase becomes very small whereas the behavior of this matrix phase becomes very soft. This configuration may be of physical interest for materials where the deformation is mainly localized in very thin areas along grains (grain boundaries, viscous films). Previous works have focused on this configuration with the help of finite-element computational methods [12, 13]. In particular, creep of a polycrystalline solid was modeled by attributing a Newtonian viscosity to the grain boundaries and a power-law creep viscosity to the grain interior. The finite-element calculations gave the flow field within the polycrystal and the macroscopic stress/strain rate behavior. At high strain rates, it was shown that the polycrystal flows according to the power-law of the grains. At low strain rates, it again flows according to this power-law, accelerated somewhat by grain boundary sliding. The transition from power-law to accelerated-power-law behavior occurs at a transitional strain rate. This transitional strain rate was calculated as well as the corresponding stress enhancement factor. It is now verified that the three phase model can predict such behaviors. Analytical results are derived for the transitional strain rate and the stress enhancement factor.

To do this, equation (A5) is evaluated when simultaneously the concentration c tends to 1 and the ratio $\bar{\mu}_2/\bar{\mu}_1$ tends to infinity. The linear case is first studied. As recalled in the Appendix, the effective modulus $\bar{\mu}$ is of the same order of magnitude as $\bar{\mu}_2$. A Taylor expansion with respect to $\bar{\mu}_2/\bar{\mu}_1$ and $c' = 1 - c$ transforms the second order equation (A5) into equation (A9). The solution $\bar{\mu}/\bar{\mu}_2$ is a function only of the product $p = (\bar{\mu}_2/\bar{\mu}_1)c'$. Variations of $\bar{\mu}/\bar{\mu}_2$ as a function p are plotted in Fig. 7. The effective modulus $\bar{\mu}$ is always located in the range $[0.475\bar{\mu}_2; \bar{\mu}_2]$. The value $0.475\bar{\mu}_2$ corresponds to the limit $p \to \infty$: the boundary region is very soft compared with the inclusion or its volume fraction is "large" enough. It is interesting to notice that this value constitutes the lowest value which may be attainable if one considers the mechanical effect of a soft boundary region within a grained material.

A usual configuration for non-linear behaviors is as follows: the grains have a strain rate sensitivity m_2 ranging from 0.2 to 0.5 and boundaries obey a Newtonian flow characterized by $m_1 = 1$. In such a case

* $\bar{\mu}_2/\bar{\mu}_1$ becomes very large for very low strain rates. $\bar{\mu}/\bar{\mu}_2$ tends towards 0.475 for a given c'. The effective constitutive law is therefore

$$\left(\frac{\sigma_0}{\sigma_c}\right) = 0.475 \times 3\bar{\mu}_2\left(\frac{\dot{\epsilon}_0}{\dot{\epsilon}_c}\right) \tag{27}$$

* $\bar{\mu}_2/\bar{\mu}_1$ becomes very low for very large strain rates. $\bar{\mu}/\bar{\mu}_2$ becomes equal to 1 for a given c'. The effective constitutive law is therefore

$$\left(\frac{\sigma_0}{\sigma_c}\right) = 3\bar{\mu}_2\left(\frac{\dot{\epsilon}_0}{\dot{\epsilon}_c}\right). \tag{28}$$

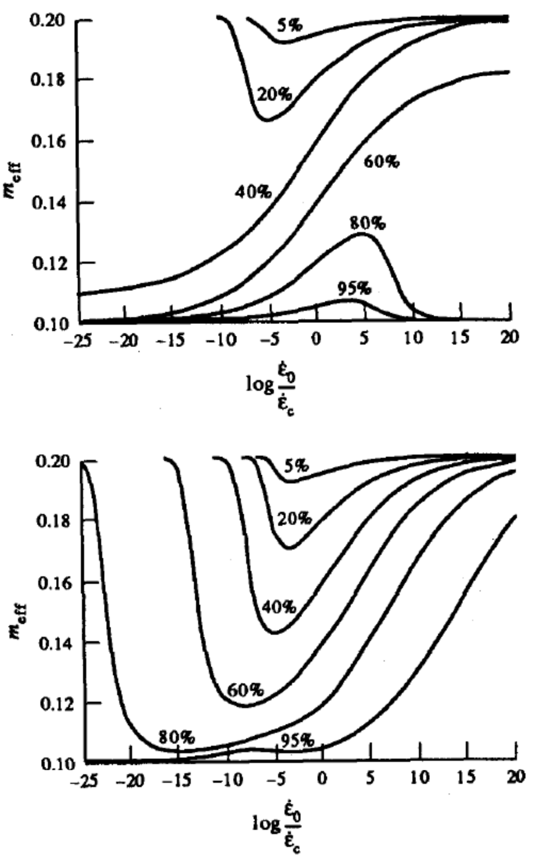


Fig. 6. The effective strain rate sensitivity vs the normalized applied strain rate. Three phase and self-consistent results for a varying inclusion volume fraction c. $m_1 = 0.2$, $m_2 = 0.1$.

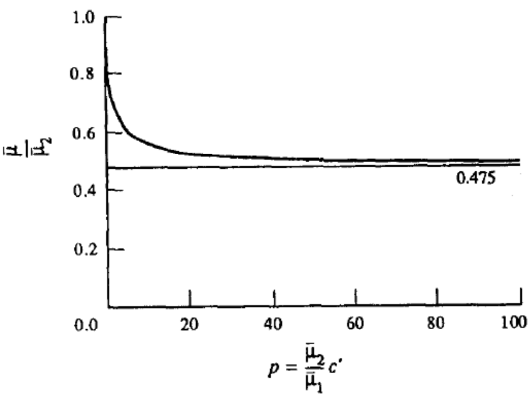


Fig. 7. Grain boundary configuration: $\bar{\mu}/\bar{\mu}_2$ vs $p = (\bar{\mu}_2/\bar{\mu}_1)c'$.

The value $p \approx 1$ may be considered as characterizing a transition from equations (27) to (28). Furthermore, $p \approx 1$ may be written $\bar{\mu}_2/\bar{\mu}_1 \approx 1/(1-c)$. The transition region corresponds then to the following strain rate

$$\dot{\epsilon}_0 \approx \dot{\epsilon}_2 \approx \left(\frac{1}{1-c}\right)^{1/(m_2-1)} \dot{\epsilon}_c \approx \left(3\frac{T_b}{R}\right)^{1/(1-m_2)} \dot{\epsilon}_c \quad (29)$$

where T_b is the thickness of the boundary region and R is the radius of the grain. Similar results were obtained by Crossman and Ashby [11] and Gharemani [12] by 2D finite-element simulations. These authors had defined a stress enhancement factor f which quantify the transition from equations (27) to (28). Our 3D analytical model leads to $f = 1/0.475 \approx 2.11$ which is clearly higher than the values ranging from 1.2 to 1.3 given by 2D finite-element simulations. This difference can be explained by different reasons:

(a) finite-element computations were performed in a finite range of strain rates and asymptotic behaviors were not always obtained
(b) the present model is a 3D one, whereas the previous results were obtained by 2D simulations.

In order to evaluate the order of approximation involved in the asymptotic constitutive laws of equations (27) and (28), numerical simulations were performed too. Results are reported in Fig. 8 for $c = 0.95$, $c = 0.97$ and $c = 0.99$, corresponding to $3(T_b/R) = 0.05$, 0.03 and 0.01. The strain rate sensitivity for the grains was chosen equal to $m_2 = 0.33$. The part of the curves for $\dot{\epsilon}_0 > \dot{\epsilon}_c$ is reported but is not realistic, because it corresponds to a boundary which would be stiffer than the matrix. The curves display a local maximum which is clearly related to the transition region analytically predicted. The value of this maximum is governed by equation (29). The values of m_{eff} tend again towards $m_1 = 1$ when very low strain rates are applied. The previous analytical derivations are then justified only around the transition region.

4. CONCLUSION

The study was devoted to the prediction of the constitutive behavior of matrix-inclusion composites. Both phases were characterized by power-law constitutive equations. The specific morphology was taken into account by using a three phase model. The power-law constitutive equations were linearized. The so-defined moduli are not constant but are functions of the average strain rate in each phase. It is then possible to use the classical linear solution of the three phase model. If both phases have the same strain rate sensitivity (the same power-law exponent), the effective behavior of the composite is characterized by an effective prefactor. If not, the effective behavior is characterized by an effective strain rate sensitivity which is a function of the macroscopic strain rate and of the volume fraction of the phases.

The results are quite different in both cases, compared to other models which do not refer to the matrix inclusion type morphology, like the self-consistent model. The self-consistent model is mainly characterized by "quasi-percolation thresholds". The present three phase model shows that the behavior of the composite is mainly governed by the behavior of the matrix. In particular, the effective power-law exponent is always one of the matrix for very low and very high applied strain rates. However, the effective strain rate sensitivity is affected by the presence of the inclusion phase for intermediate applied strain rates. If the volume fraction of the inclusion phase becomes very large, the effective strain rate sensitivity tends to the one of the inclusion in a certain range of applied rates. A special transition behavior occurs in this range, which can be related to previous numerical studies concerning the accommodation of power-law creep by grain-boundary sliding. A stress enhancement factor has been analytically calculated. It appears as quite different than the previous ones obtained from finite element calculations.

It is worth noticing that assumptions were made in this study. In particular, the effective moduli in each

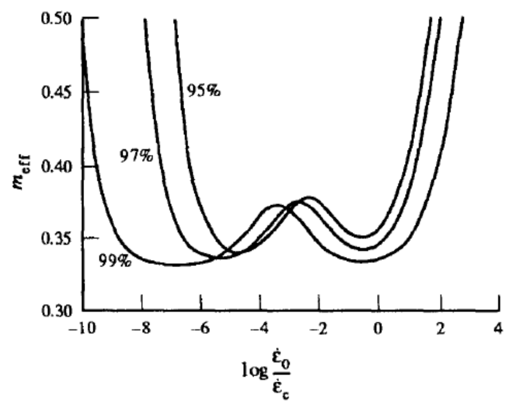


Fig. 8. Grain boundary configuration: the effective strain rate sensitivity vs the normalized applied strain rate. $m_1 = m_2 = 0.33$.

phase depend on the strain rate but they are after all assumed constant to apply the linear solution of the three phase model. It is expected that this assumption is not too crude for strain rate sensitivity values ranging from 1 to 0.1. Recent developments, such as the representative inclusion [14, 15], the "N-layered" extension of the three phase model [16] or the introduction of adequate variational principles [8] may appear as alternative schemes to better take into account the inhomogeneity of stresses and strains within each phase of non-linear composites.

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APPENDIX

The Elastic Equations of the Three Phase Model

The problem of an infinite medium constituted of a two-layered isotropic spherical inclusion embedded in a homogeneous continuum subjected to uniform conditions at infinity was first studied by Christensen and Lo [6] and later reformulated by Hervé and Zaoui [7]. The core and the shell of the two-layered inclusion represent respectively the inclusion phase and the matrix phase of the composite. The continuum represents the homogeneous equivalent medium. If $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}$ denote respectively the shear moduli of the shell, of the core and of the continuum, the average deviatoric strain tensors $\langle \underline{\epsilon} \rangle_1$ (in the "matrix phase") and $\langle \underline{\epsilon} \rangle_2$ (in the "inclusion phase") may be expressed as

$$\langle \underline{\epsilon} \rangle_1 = \frac{\bar{\mu}_2 - \bar{\mu}}{(1-c)(\bar{\mu}_2 - \bar{\mu}_1)} \langle \underline{\epsilon} \rangle = a_1 \langle \underline{\epsilon} \rangle \quad \text{(A1)}$$

$$\langle \underline{\epsilon} \rangle_2 = \frac{\bar{\mu} - \bar{\mu}_1}{c(\bar{\mu}_2 - \bar{\mu}_1)} \langle \underline{\epsilon} \rangle = a_2 \langle \underline{\epsilon} \rangle \quad \text{(A2)}$$

c denotes the volume fraction of the core. $\langle \epsilon \rangle$ is the average deviatoric strain tensor over the two-layered inclusion. If $u_0 = \epsilon_0 \cdot x$ denotes the prescribed displacement field at infinity, $\langle \underline{\epsilon} \rangle = \underline{\epsilon}_0$.

Similar relations are given for the deviatoric stress tensors

$$\langle \underline{s} \rangle_1 = \frac{\bar{\mu}_1(\bar{\mu}_2 - \bar{\mu})}{(1-c)\bar{\mu}(\bar{\mu}_2 - \bar{\mu}_1)} \langle \underline{s} \rangle = b_1 \langle \underline{s} \rangle \quad \text{(A3)}$$

$$\langle \underline{s} \rangle_2 = \frac{\bar{\mu}_2(\bar{\mu} - \bar{\mu}_1)}{c\bar{\mu}(\bar{\mu}_2 - \bar{\mu}_1)} \langle \underline{s} \rangle = b_2 \langle \underline{s} \rangle. \quad \text{(A4)}$$

It is worth noticing that $ca_2 + (1-c)a_1 = 1$ and $cb_2 + (1-c)b_1 = 1$.

Finally, $\bar{\mu}/\bar{\mu}_1$ appears as the positive root of a second order equation

$$A\left(\frac{\bar{\mu}}{\bar{\mu}_1}\right)^2 + B\left(\frac{\bar{\mu}}{\bar{\mu}_1}\right) + C = 0 \quad \text{(A5)}$$

with

$$\begin{aligned} A = {} & 228c^{10/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)^2 - 25c^{7/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)\left(57\frac{\bar{\mu}_2}{\bar{\mu}_1} + 54\right) \\ & + 126c^{5/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right) - 75c\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right) \\ & \times\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right) + 6\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right)\left(2\frac{\bar{\mu}_2}{\bar{\mu}_1} + 3\right) \quad \text{(A6)} \end{aligned}$$

$$\begin{aligned} B = {} & 114c^{10/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)^2 + 50c^{7/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)\left(57\frac{\bar{\mu}_2}{\bar{\mu}_1} + 54\right) \\ & - 252c^{5/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right) + \tfrac{375}{4}c\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right) \\ & \times\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right) + \frac{9}{8}\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right)\left(2\frac{\bar{\mu}_2}{\bar{\mu}_1} + 3\right) \quad \text{(A7)} \end{aligned}$$

$$\begin{aligned} C = {} & -342c^{10/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)^2 - 25c^{7/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)\left(57\frac{\bar{\mu}_2}{\bar{\mu}_1} + 54\right) \\ & + 126c^{5/3}\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right)\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right) - \tfrac{675}{8}c\left(\frac{\bar{\mu}_2}{\bar{\mu}_1} - 1\right) \\ & \times\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right) - \frac{57}{8}\left(19\frac{\bar{\mu}_2}{\bar{\mu}_1} + 16\right)\left(2\frac{\bar{\mu}_2}{\bar{\mu}_1} + 3\right). \quad \text{(A8)} \end{aligned}$$

If the volume fraction of the inclusion phase $c = 1 - c'$ tends towards one and when its shear modulus tends towards infinity, the ratio $\bar{\mu}/\bar{\mu}_1$ and $\bar{\mu}_2/\bar{\mu}_1$ are of the same order of magnitude. The parameter λ defined by $\bar{\mu}/\bar{\mu}_1 = \lambda(\bar{\mu}_2/\bar{\mu}_1)$ is the positive root of the following equation

$$\left(336 + 560c'\frac{\bar{\mu}_2}{\bar{\mu}_1}\right)\lambda^2 + \left(63 - 266c'\frac{\bar{\mu}_2}{\bar{\mu}_1}\right)\lambda - 399 = 0. \quad \text{(A9)}$$